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HOMOGENEOUS SOLUTIONS OF TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY FOR A RECTANGULAR REGION OF A COSSERAT MEDIUM

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The theory of media whose properties are derived from the variational principle termed "euclidean action" by E. and F. Cosserat is developed in the book by Appell [1]. The arguments of the integrand in the mathematical expression of this principle are entirely determined by the geometry of the space. In the general statical case, 21 independent kinematical elements occur in this function for euclidean space. Certain contemporary authors, [2, 3] and others reduce the number of arguments to 14 by imposing differential relations among some of them. In particular, the components of the displacement vector and the components of the curl of the displacement vector are examined. Proceeding from the last precondition of [4], the basic apparatus for solution of the two-dimensional problem of the theory of elasticity is given and used in the present paper.

1. The solution of the two-dimensional problem with the aid of the Airy and Mindlin stress functions. The components of the stresses and couple stresses may be expressed with the aid of two stress functions in the following way:

$$\sigma_x = \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y}, \quad \mu_x = \frac{\partial \psi}{\partial x}, \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}, \quad \mu_y = \frac{\partial \psi}{\partial y}$$

$$\tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2}, \quad \tau_{yx} = -\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} \tag{1.1}$$

where the function φ is the usual Airy stress function and ψ is the stress function introduced by Mindlin.

The stress functions must satisfy the following differential equations [4]:

$$\nabla^4\varphi = 0,\ \nabla^2\psi - l^2\nabla^4\psi = 0 \quad (l^2 = B / G) \tag{1.2}$$

where l is a special elastic constant for a Cosserat material and has the dimensions of length, G is the shear modulus, and B is a modulus relating the curvature to the couple stress components according to the equations

$$\frac{\partial\omega_z}{\partial x} = \frac{1}{4B}\mu_x, \quad \frac{\partial\omega_z}{\partial y} = \frac{1}{4B}\mu_y \tag{1.3}$$

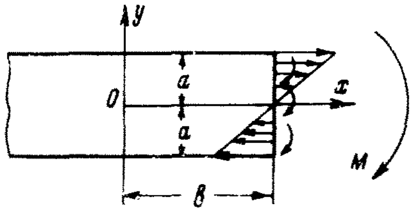
Fig. 1

We shall examine the rectangular region shown in Fig. 1. We seek homogeneous solutions for this region in which the boundaries $y = \pm a$ are free of both stresses and couple stresses. There are two types of solutions:

1) Homogeneous solutions having force symmetry about the x-axis; and

2) homogeneous solutions having force quantities which are antisymmetric with respect to the x-axis.

In what follows, we shall present homogeneous solutions of the second type.

2. Homogeneous solutions for a rectangular region, two parallel sides of which are free of stresses and couple stresses. The desired solution of the second type (antisymmetric about the x-axis) can be constructed with the aid of the stress functions

$$\varphi = \operatorname{ch}\alpha x\,(C_1 \sin\alpha y + C_2 y \cos\alpha y) \tag{2.1}$$

$$\psi = Bl^2 \operatorname{sh}\alpha x \cos\beta y + 4(1-\nu)\,l^2\alpha C_2 \operatorname{sh}\alpha x \cos\alpha y$$

$$\alpha^2 = \beta^2 + l^{-2} \tag{2.2}$$

where C_1, C_2 and B are arbitrary constants. From the condition that the stresses and couple stresses are zero on the boundaries $y = \pm a$ the following transcendental equation is obtained:

$$\frac{\alpha a}{\operatorname{tg}\alpha a} + \alpha a \operatorname{tg}\alpha a - 1 - 4(1-\nu)\,l^2\alpha^2\left(\frac{\alpha}{\beta}\frac{\operatorname{tg}\alpha a}{\operatorname{tg}\beta\alpha} - 1\right) = 0 \tag{2.3}$$

This transcendental equation has an infinite number of roots. The components of the stresses and couple stresses are expressed by the equations

$$\sigma_x = \sum_i C_i \operatorname{ch}\alpha_i x\left[\left(\frac{\alpha_i^2 a}{\operatorname{tg}\alpha_i a} - 2\alpha_i\right)\sin\alpha_i y - \alpha_i^2 y\cos\alpha_i y - \right.$$
$$\left. - 4(1-\nu)\,\alpha_i^3 l^2\left(\frac{\sin\alpha_i a}{\sin\beta_i a}\sin\beta_i y - \sin\alpha_i y\right)\right]$$

$$\sigma_y = \sum_i C_i \operatorname{ch}\alpha_i x\left[-\frac{\alpha_i^2 a}{\operatorname{tg}\alpha_i a}\sin\alpha_i y + \alpha_i^2 y\cos\alpha_i y + \right.$$
$$\left. + 4(1-\nu)\,\alpha_i^3 l^2\left(\frac{\sin\alpha_i a}{\sin\beta_i a}\sin\beta_i y - \sin\alpha_i y\right)\right]$$

$$\tau_{xy} = \sum_i C_i \operatorname{sh}\alpha_i x\left[\left(\frac{\alpha_i^2 a}{\operatorname{tg}\alpha_i a} - \alpha_i\right)\cos\alpha_i y + \alpha_i^2 y\sin\alpha_i y - \right.$$
$$\left. - 4(1-\nu)\,l^2\alpha_i^2\left(\beta_i\frac{\sin\alpha_i a}{\sin\beta_i a}\cos\beta_i y - \alpha_i\cos\alpha_i y\right)\right]$$

$$\tau_{yx} = \sum_i C_i \operatorname{sh}\alpha_i x \left[\left(\frac{\alpha_i^2 a}{\operatorname{th}\alpha_i a} - \alpha_i\right)\cos\alpha_i y + \alpha_i^2 y \sin\alpha_i y - \right.$$

$$\left. - 4(1-\nu)\, l^2 \alpha_i^3 \left(\frac{\alpha_i}{\beta_i}\frac{\sin\alpha_i a}{\sin\beta_i a}\cos\beta_i y - \cos\alpha_i y\right)\right]$$

$$\mu_x = 4(1-\nu)\, l^2 \sum_i C_i \alpha_i^2 \operatorname{ch}\alpha_i x \left(-\frac{\alpha_i}{\beta_i}\frac{\sin\alpha_i a}{\sin\beta_i a}\cos\beta_i y + \cos\alpha_i y\right)$$

$$\mu_y = 4(1-\nu)\, l^2 \sum_i C_i \alpha_i^2 \operatorname{sh}\alpha_i x \left(\frac{\sin\alpha_i a}{\sin\beta_i a}\sin\beta_i y - \sin\alpha_i y\right) \qquad (2.4)$$

The roots of the transcendental equation (2.3) depend on the elastic constant l for a Cosserat medium (it occurs in the expression for α) and on Poisson's ratio ν. For $l \to \infty$ the couple stresses have the greatest effect; here in the limit $\beta = \alpha$ and the transcendental equation (2.3) reduces to the form

$$\frac{2\alpha a}{\sin 2\alpha a} = -\frac{3-2\nu}{1-2\nu} \quad \left(\frac{2\alpha a}{\sin 2\alpha a} = -6 \quad \text{for} \quad \nu = 0.3\right) \qquad (2.5)$$

The other extreme case, when $l = 0$, leads to the transcendental equation for the symmetric (classical) theory of elasticity without couple stresses [5], namely

$$\frac{2\alpha a}{\sin 2\alpha a} = 1 \qquad (2.6)$$

This equation has only complex roots (except for the root at zero). The first two roots of the transcendental equation (2.5) for $\nu = 0.3$ are real and the remaining ones are complex. We give the first twelve roots below.

$$2\alpha_i a = 2r_k + 2t_k i = x + iy$$

	1	2	3	4
$x=$	3.835	5.226	10.863197	17.172427
$y=$	0	0	—1.2104393	—1.7187154
	5	6	7	8
$x=$	23.472110	29.766339	36.058627	42.348509
$y=$	—2.0446282	— 2. 2875990	—2.4820034	—2.6442697
	9	10	11	12
$x=$	48.637078	54.924699	61.211611	67.497976
$y=$	—2. 7836203	— 2.9057768	—3.0145427	—3.1125788

The transcendental equation (2.3) has complex roots and, therefore, the solution given by Eqs. (2.4) are in complex form. The transformation to the real form of the solution is accomplished by considering that the conjugate roots also satisfy the transcendental equation. By combining the two variants of the expressions (2.4) for the system of complex conjugate roots and introducing complex conjugate arbitrary constants, we obtain the desired solutions in real form. Thus, to each complex root there correspond two arbitrary real constants.

3. Representation of the solution in real form; the boundary value problem for two loaded edges of a rectangular region. We represent the solution in real form corresponding to the transcendental equation (2.5). The stress functions for this case are

$$\varphi = 2a \sum_k \left[a_k R_k(x, y) - b_k S_k(x, y)\right]$$
$$\psi = -4(1-\nu)\, a \sum_k \left[a_k P_k(x, y) - b_k Q_k(x, y)\right] \tag{3.1}$$

where a_k, b_k are arbitrary real constants corresponding to the kth root of the transcendental equation (2.5), and $R_k(x, y)$, $S_k(x, y)$, $P_k(x, y)$, $Q_k(x, y)$ are real functions of their arguments corresponding to the same kth root. We now give the expressions for these functions

$$R_k(x, y) = -A_k(\mathrm{Y}_1X_1 - \mathrm{Y}_2X_2) - B_k(\mathrm{Y}_1X_2 + \mathrm{Y}_2X_1) + \\ + (y/a)\,\mathrm{Y}_3X_1\,(y/a)\,\mathrm{Y}_4X_2$$

$$S_k(x, y) = B_k(\mathrm{Y}_1X_1 - \mathrm{Y}_2X_2) - A_k(\mathrm{Y}_1X_2 + \mathrm{Y}_2X_1) + \\ + (y/a)\,\mathrm{Y}_3X_2 - (y/a)\,\mathrm{Y}_4X_1$$

$$P_k(x, y) = (A_k + C_k)(\mathrm{Y}_3X_3 + \mathrm{Y}_4X_4) + (B_k + D_k)(\mathrm{Y}_3X_4 - \mathrm{Y}_4X_3) - \\ - (y/a)\,\mathrm{Y}_1X_3 - (y/a)\,\mathrm{Y}_2X_4 \tag{3.2}$$

$$Q_k(x, y) = (A_k + C_k)(\mathrm{Y}_3X_4 - \mathrm{Y}_4X_3) - (B_k + D_k)(\mathrm{Y}_4X_4 + \mathrm{Y}_3X_3) + \\ + (y/a)\,\mathrm{Y}_1X_4 + (y/a)\,\mathrm{Y}_2X_3$$

where

$$A_k = \frac{\sin 2r_k}{2(\sin^2 r_k \operatorname{ch}^2 t_k + \cos^2 r_k \operatorname{sh}^2 t_k)}, \qquad C_k = \frac{r_k}{r_k^2 + t_k^2} \tag{3.3}$$

$$B_k = \frac{\operatorname{sh} 2t_k}{2(\sin^2 r_k \operatorname{ch}^2 t_k + \cos^2 r_k \operatorname{sh}^2 t_k)}, \qquad D_k = \frac{t_k}{r_k^2 + t_k^2}$$

$$X_1 = \operatorname{ch}\frac{r_k x}{a}\cos\frac{t_k x}{a}, \quad X_2 = \operatorname{sh}\frac{r_k x}{a}\sin\frac{t_k x}{a}, \quad X_3 = \operatorname{sh}\frac{r_k x}{a}\cos\frac{t_k x}{a}$$

$$X_4 = \operatorname{ch}\frac{r_k x}{a}\sin\frac{t_k x}{a}, \quad Y_1 = \sin\frac{r_k y}{a}\operatorname{ch}\frac{t_k y}{a}, \quad Y_2 = \cos\frac{r_k y}{a}\operatorname{sh}\frac{t_k y}{a}$$

$$Y_3 = \cos\frac{r_k y}{a}\operatorname{ch}\frac{t_k y}{a}, \quad Y_4 = \sin\frac{r_k y}{a}\operatorname{sh}\frac{t_k y}{a}$$

The stress and couple stress components are expressed by the following equations:

$$\sigma_x = 2a\sum_k\left(a_k\frac{\partial^2 R_k}{\partial y^2} - b_k\frac{\partial^2 S_k}{\partial y^2}\right) + 4(1-\nu)\,a\sum_k\left(a_k\frac{\partial^2 P_k}{\partial x\partial y} - b_k\frac{\partial^2 Q_k}{\partial x\partial y}\right)$$

$$\sigma_y = 2a\sum_k\left(a_k\frac{\partial^2 R_k}{\partial x^2} - b_k\frac{\partial^2 S_k}{\partial x^2}\right) - 4(1-\nu)\,a\sum_k\left(a_k\frac{\partial^2 P_k}{\partial x\partial y} - b_k\frac{\partial^2 Q_k}{\partial x\partial y}\right)$$

$$\tau_{xy} = -2a\sum_k\left(a_k\frac{\partial^2 R_k}{\partial x\partial y} - b_k\frac{\partial^2 S_k}{\partial x\partial y}\right) + 4(1-\nu)\,a\sum_k\left(a_k\frac{\partial^2 P_k}{\partial y^2} - b_k\frac{\partial^2 Q_k}{\partial y^2}\right)$$

$$\tau_{yx} = -2a\sum_k\left(a_k\frac{\partial^2 R_k}{\partial x\partial y} - b_k\frac{\partial^2 S_k}{\partial x\partial y}\right) - 4(1-\nu)\,a\sum_k\left(a_k\frac{\partial^2 P_k}{\partial x^2} - b_k\frac{\partial^2 Q_k}{\partial x^2}\right)$$

$$\mu_x = -4(1-\nu)\,a\sum_k\left(a_k\frac{\partial P_k}{\partial x} - b_k\frac{\partial Q_k}{\partial x}\right)$$

$$\mu_y = -4(1-\nu)\,a\sum_k\left(a_k\frac{\partial P_k}{\partial y} - b_k\frac{\partial Q_k}{\partial y}\right) \tag{3.4}$$

The arbitrary constants a_k and b_k are determined from the following boundary conditions: for $x = b$ (force symmetry with respect to the y-axis)

$$\sigma_x = \sigma_x^* (y), \quad \tau_{xy} = \tau_{xy}^* (y), \quad \mu_x = \mu_x^*(y)$$

where these specified functions of y must satisfy the condition of antisymmetry with respect to the x-axis.

A quite simple solution is obtained for the case of pure bending in plane strain or generalized plane stress. These cases are also homogeneous solutions, inasmuch as two parallel edges of the region remain free. The case of pure bending in plane strain is obtained with the aid of the following two stress functions:

$$\varphi = \frac{D}{6} y^3, \quad \psi = -2(1-\nu)\, l^2 D x \tag{3.5}$$

The components of the stresses and couple stresses are then

$$\sigma_x = Dy, \quad \sigma_y = 0, \quad \tau_{yx} = 0, \quad \tau_{xy} = 0$$
$$\mu_y = 0, \quad \mu_x = -2(1-\nu)\, l^2 D \tag{3.6}$$

The couple stresses are uniformly distributed with depth (along y). The constant D is determined by equating the moment formed from the stresses and couple stresses to the bending moment M

$$M = \int_{-a}^{+a} \sigma_x y\, dy - \int_{-a}^{+a} \mu_x\, dy = D \frac{2}{3} a^3 \left[1 + 6(1-\nu)\frac{l^2}{a^2}\right] \tag{3.7}$$

whence

$$D = \frac{M}{{}^2/_3 a^3 \left[1 + 6(1-\nu)\, l^2/a^2\right]} \tag{3.8}$$

Determining the displacement by integration, we obtain the equation for the deflection curve in the following form:

$$v\,|_{y=0} = -\frac{M(1-\nu^2)}{2EJ_z(1 + L^2/a^2)} x^2, \qquad L^2 = 6(1-\nu)\, l^2 \tag{3.9}$$

Equation (3.9) shows that in the presence of couple stresses the flexural stiffness is increased to $(1 + L^2/a^2)$ times the ordinary flexural stiffness for a medium corresponding to the classical theory.

Experiments carried out by A.I. Ladatkin (graduate student in the department of structural mechanics at the Leningrad Academy for Wood Technology) on steel strips indicate practically no increase in flexural stiffness. Thus, for steel, the elastic characteristic l for a Cosserat medium is very small and it is practically impossible to observe the couple stresses. This result is in complete accord with the observation in [6], referring to [7], that for engineering metals the elastic constant l for a Cosserat medium is of the order $l \approx 0.1$ mm.

The work presented above shows that homogeneous solutions can be constructed for a rectangular region of a Cosserat material. These homogeneous solutions are useful in considering boundary value problems for rectangular regions loaded on the edges by distributed forces and couples.

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STABILITY OF PLANE-PARALLEL CONVECTIVE MOTION WITH RESPECT TO SPATIAL PERTURBATIONS

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The Squire transformation, known in the theory of hydrodynamic stability [1, 2], makes it possible to reduce the problem of a plane-parallel isothermal motion with respect to spatial perturbations to a problem of plane perturbations.

Formulas derived by Squire for the transformation of the Reynolds and the wave numbers allow the derivation of complete information on stability from a solution of the two-dimensional boundary value problem of Orr-Sommerfeld. It was found that plane perturbations are more dangerous because smaller (as compared to spatial perturbations) critical Reynolds numbers correspond to them.

The problem becomes more complicated in the case of a nonisothermal plane-parallel flow. The stability of a plane Poisuille flow between horizontal parallel planes heated to different temperatures was considered in [3]. A transformation similar to that of Squire is applicable in this case also, but contrary to the isothermal case, the spatial perturbations at certain specific values of parameters are here relatively more dangerous.

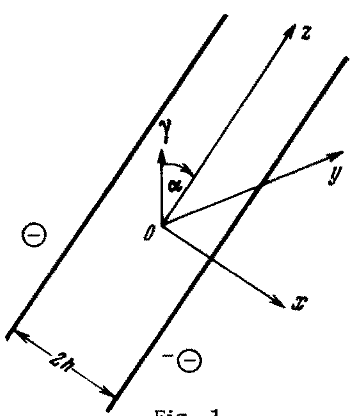

Fig. 1

The stability relative to spatial perturbations of free stationary convective motions (due to temperature nonuniformity) between infinite parallel planes, heated to different temperatures and arbitrarily orientated in the gravitational field, is considered below (Fig. 1).

Transformations of the Grashof and wave numbers, and of the angle of the layer with the vertical are derived, thus reducing the problem of stability with respect to normal spatial perturbations to the equivalent problem of plane perturbations. As the result of these transformations together with stability investigations with respect to plane perturbations [4], diagrams of convective flow stability with respect to three-dimensional perturbations were obtained.